

Distance and Similarity Measures for Soft Sets

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ABSTRACT. In [P. Majumdar, S. K. Samanta, *Similarity measure of soft sets*, New Mathematics and Natural Computation 4(1)(2008) 1-12], the authors use matrix representation based distances of soft sets to introduce matching function and distance based similarity measures. We first give counterexamples to show that their Definition 2.7 and Lemma 3.5(3) contain errors, then improve their Lemma 4.4 making it a corollary of our result. The fundamental assumption of [7] has been shown to be flawed. This motivates us to introduce set operations based measures. We present a case (Example 28) where Majumdar-Samanta similarity measure produces an erroneous result but the measure proposed herein decides correctly. Several properties of the new measures have been presented and finally the new similarity measures have been applied to the problem of financial diagnosis of firms.

Keywords: Applied soft sets; Similarity measure; Distance measure; Financial diagnosis; Similarity based decision making;

1. Introduction

In 1999, D. Molodtsov [9], introduced the notion of a soft set as a collection of approximate descriptions of an object. This initial description of the object has an approximate nature, and we do not need to introduce the notion of exact solution. The absence of any restrictions on the approximate description in soft sets make this theory very convenient and easily applicable in practice. Applications of soft sets in areas ranging from decision problems to texture classification, have surged in recent years [5, 6, 10, 13, 14].

Similarity measures quantify the extent to which different patterns, signals, images or sets are alike. Such measures are used extensively in the application of fuzzy sets, intuitionistic fuzzy set and vague sets to the problems of pattern recognition, signal detection, medical diagnosis and security verification systems. That is why several researchers have studied the problem of similarity measurement between fuzzy sets [3], intuitionistic fuzzy sets (IFSs) and vague sets. Ground breaking work for introducing similarity measure of soft sets was presented by Majumdar and Samanta in [7]. Their work uses matrix representation based distances of soft sets to introduce similarity measures. In this paper, we propose new similarity measures using set theoretic operations, besides showing how the earlier similarity measures of Majumdar and Samanta are inappropriate. We also present an application of the proposed measures of similarity in the area of automated financial analysis.

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This paper is organized as follows: in Section 2, requisite preliminary notions from Soft Set Theory have been presented. Section 3 comprises some counterexamples to show that some claims in [7] are not correct. At the end of this section we also improve and generalize Lemma 4.4 of [7]. In Section 4 we give our motivation and rationale to introduce set operations based distance and similarity measures. Section 5 introduces the notion of set operation based distance between soft sets and some of its weaker forms. In Section 6 similarity measures have been defined. Finally Section 7 is the application of new similarity measures to the problem of financial diagnosis of firms.

2. Preliminaries

A pair (F, A) is called a soft set [9] over X , where F is a mapping given by $F : A \rightarrow P(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X . For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε -approximate elements of the soft set (F, A) . Clearly a soft set is not a set in ordinary sense.

Definition 1. [2] Let X be a universe and E a set of attributes. Then the pair (X, E) , called a soft space, is the collection of all soft sets on X with attributes from E .

Definition 2. [11] For two soft sets (F, A) and (G, B) over X , we say that (F, A) is a soft subset of (G, B) , if

- (i) $A \subseteq B$, and
- (ii) $\forall \varepsilon \in A, F(\varepsilon) \subseteq G(\varepsilon)$.

We write $(F, A) \subseteq (G, B)$. (F, A) is said to be a soft super set of (G, B) , if (G, B) is a soft subset of (F, A) . We denote it by $(F, A) \supseteq (G, B)$.

Definition 3. [8] Union of two soft sets (F, A) and (G, B) over a common universe X is the soft set (H, C) , where $C = A \cup B$, and $\forall \varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & \text{if } \varepsilon \in A - B, \\ G(\varepsilon), & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \cup G(\varepsilon), & \text{if } \varepsilon \in A \cap B. \end{cases}$$

We write $(F, A) \cup (G, B) = (H, C)$.

Definition 4. [1] Let (F, A) and (G, B) be two soft sets over X with $A \cap B \neq \phi$. Restricted intersection of two soft sets (F, A) and (G, B) is a soft set (H, C) , where $C = A \cap B$, and $\forall \varepsilon \in C, H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$. We write $(F, A) \cap (G, B) = (H, C)$.

Definition 5. [7] The complement of a soft set (F, A) is denoted by $(F, A)^c$, and is defined by $(F, A)^c = (F^c, A)$, where $F^c : A \rightarrow P(X)$ is a mapping given by $F^c(\varepsilon) = (F(\varepsilon))^c, \forall \varepsilon \in A$.

In the sequel we shall denote the absolute null and absolute whole soft sets in a soft space (X, E) as (F_ϕ, E) and (F_X, E) , respectively. These have been defined in [1] as:

$$\begin{aligned} (F_\phi, E) &= \{\varepsilon = \phi \mid \forall \varepsilon \in E\}, \\ (F_X, E) &= \{\varepsilon = X \mid \forall \varepsilon \in E\}. \end{aligned}$$

3. Counterexamples

Recently Majumdar and Samanta [7] have written the ground breaking paper on similarity measures of soft sets. In this section we first give counterexamples to show that their Definition 2.7 and Lemma 3.5(3) contain some errors. We, then, improve Lemma 4.4 [7] and make it a corollary of our result.

A matching function based similarity measure has been defined in [7] as:

Definition 6. [7] If $E_1 = E_2$, then similarity between (F_1, E_1) and (F_2, E_2) is defined by

$$S(F_1, F_2) = \frac{\sum_i \vec{F}_1(e_i) \cdot \vec{F}_2(e_i)}{\sum_i [\vec{F}_1(e_i)^2 \vee \vec{F}_2(e_i)^2]}. \quad (\text{I})$$

If $E_1 \neq E_2$ and $E = E_1 \cap E_2 \neq \emptyset$, then we first define $\vec{F}_1(e) = \underline{0}$ for $e \in E_2 \setminus E$ and $\vec{F}_2(f) = \underline{0}$ for $f \in E_1 \setminus E$. Then $S(F_1, F_2)$ is defined by formula (I).

Lemma 7. (Lemma 3.5 [7]) Let (F_1, E_1) and (F_2, E_2) be two soft sets over the same finite universe U . Then the following hold:

- (1) $S(F_1, F_2) = S(F_2, F_1)$
- (2) $0 \leq S(F_1, F_2) \leq 1$
- (3) $S(F_1, F_1) = 1$

Our next example shows that claim (3) of Lemma 7 is incorrect:

Example 8. Let $X = \{a, b, c\}$ and $E = \{e_1, e_2, e_3\}$. We choose soft set (F_1, E) as:

$$(F_1, E) = \{e_1 = \{\}, e_2 = \{\}, e_3 = \{\}\}$$

Then using (I) we get

$$S(F_1, F_1) = \frac{0}{0}.$$

Majumdar and Samanta have defined following distances between soft sets as four distinct notions:

Definition 9. [7] For two soft sets (\hat{F}_1, E) and (\hat{F}_2, E) we define the mean Hamming distance $D^s(\hat{F}_1, \hat{F}_2)$ between soft sets as:

$$D^s(\hat{F}_1, \hat{F}_2) = \frac{1}{m} \left\{ \sum_{i=1}^m \sum_{j=1}^n \left| \hat{F}_1(e_i)(x_j) - \hat{F}_2(e_i)(x_j) \right| \right\},$$

the normalized Hamming distance $L^s(\hat{F}_1, \hat{F}_2)$ as:

$$L^s(\hat{F}_1, \hat{F}_2) = \frac{1}{m \cdot n} \left\{ \sum_{i=1}^m \sum_{j=1}^n \left| \hat{F}_1(e_i)(x_j) - \hat{F}_2(e_i)(x_j) \right| \right\},$$

the Euclidean distance $E^s(\hat{F}_1, \hat{F}_2)$ as:

$$E^s(\hat{F}_1, \hat{F}_2) = \sqrt{\frac{1}{m} \sum_{i=1}^m \sum_{j=1}^n \left(\hat{F}_1(e_i)(x_j) - \hat{F}_2(e_i)(x_j) \right)^2},$$

the normalized Euclidean distance as:

$$Q^s(\hat{F}_1, \hat{F}_2) = \sqrt{\frac{1}{m n} \sum_{i=1}^m \sum_{j=1}^n \left(\hat{F}_1(e_i)(x_j) - \hat{F}_2(e_i)(x_j) \right)^2}.$$

Remark 10. Majumdar and Samanta's observation (also used in their proof of Lemma 4.4 [7]) that

$$|F(e_i)(x_j) - G(e_i)(x_j)| \leq 1$$

is inaccurate. The quantity $|F(e_i)(x_j) - G(e_i)(x_j)|$ is either 0 or 1, only. Consequently, the term $(F(e_i)(x_j) - G(e_i)(x_j))^2$, used for defining distances E^s and Q^s , comes out to be identical with

$$|F(e_i)(x_j) - G(e_i)(x_j)|.$$

This renders E^s and Q^s as mere square roots of D^s and L^s , respectively. Symbolically we write:

$$E^s = \sqrt{D^s} \text{ and } Q^s = \sqrt{L^s}.$$

Hence the four distances of Majumdar and Samanta are not distinct, rather they are only two distances.

In the sequel the cardinality of a set A is denoted as $\|A\|$. We now present the main result of this section as:

Theorem 11. Let $m = \|E\|$, $n = \|X\|$. Then for any two soft sets (F_1, E) and (F_2, E) we have

- (1) $D^s(F_1, F_2) \in \left\{ \frac{k}{m} \mid k = 0, 1, 2, \dots, mn \right\}$,
- (2) $L^s(F_1, F_2) \in \left\{ \frac{k}{mn} \mid k = 0, 1, 2, \dots, mn \right\}$,
- (3) $E^s(F_1, F_2) \in \left\{ \sqrt{\frac{k}{m}} \mid k = 0, 1, 2, \dots, mn \right\}$,
- (4) $Q^s(F_1, F_2) \in \left\{ \sqrt{\frac{k}{mn}} \mid k = 0, 1, 2, \dots, mn \right\}$.

Proof. (1) The smallest and the largest distances are given as

$$D^s((F_\phi, E), (F_\phi, E)) = \frac{1}{m} \left\{ \sum_{i=1}^m \sum_{j=1}^n |F_\phi(e_i)(x_j) - F_\phi(e_i)(x_j)| \right\} = 0. \quad (\text{II})$$

$$\begin{aligned} D^s((F_\phi, E), (F_X, E)) &= \frac{1}{m} \left\{ \sum_{i=1}^m \sum_{j=1}^n |F_\phi(e_i)(x_j) - F_X(e_i)(x_j)| \right\} \\ &= \frac{1}{m} \left\{ \sum_{i=1}^m \sum_{j=1}^n |0 - 1| \right\}, \text{ since } F_\phi(e_i)(x_j) = 0 \text{ and } F_X(e_i)(x_j) = 1 \forall i, j. \\ &= \frac{1}{m} \underbrace{(|0 - 1| + |0 - 1| + \dots + |0 - 1|)}_{mn \text{ times}} = \frac{mn}{m} = n. \end{aligned} \quad (\text{III})$$

Furthermore, suppose the arrangement of entries in matrix representation of two arbitrary soft sets (F_1, E) and (F_2, E) is such that the term $|F_1(e_i)(x_j) - F_2(e_i)(x_j)|$ evaluates to 1, k times. Then, we can re-arrange the terms in expansion of $D^s(F_1, F_2)$ to get

$$\begin{aligned} D^s(F_1, F_2) &= \frac{1}{m} \left(\underbrace{(|0 - 1| + |1 - 0| + |0 - 1| + \dots + |1 - 0|)}_{k \text{ times}} + \right. \\ &\quad \left. \underbrace{(|0 - 0| + |1 - 1| + |0 - 0| + \dots + |1 - 1|)}_{mn - k \text{ times}} \right) \\ &= \frac{1}{m} (k + 0) = \frac{k}{m}. \end{aligned} \quad (\text{IV})$$

By (1), (1) and (1) we have $D^s(F_1, F_2) \in \left\{ \frac{k}{m} \mid k = 0, 1, 2, \dots, mn \right\}$.

(2) Note that $L^s(F_1, F_2) = \frac{1}{n} D^s(F_1, F_2)$. The result now follows immediately from (1).

(3), (4) Follow immediately by Remark 10 and (1) and (2). ■

Corollary 12. (Lemma 4.4 [7]) Let $m = \|E\|, n = \|X\|$. Then for any two soft sets (F_1, E) and (F_2, E) we have

- (1) $D^s(F_1, F_2) \leq n$,
- (2) $L^s(F_1, F_2) \leq 1$,
- (3) $E^s(F_1, F_2) \leq \sqrt{n}$,
- (4) $Q^s(F_1, F_2) \leq 1$.

4. Motivation for Introducing New Distance and Similarity Measures

We first define the notion of soft space:

Definition 13. Let X be a universe and E a set of attributes. Then the pair (X, E) , called a soft space, is the collection of all soft sets on X with attributes from E .

The work of Majumdar and Samanta depends solely upon the tacit assumption that matrix representation of soft sets is a suitable representation. We now discuss the validity of this assumption.

Tabular representation of a soft set was first proposed by Maji, Biswas and Roy in [8]. This representation readily lends itself to become Majumdar and Samanta's matrix representation as given in [7]. Hence, in the sequel, we shall use the words 'table representation' and 'matrix representation' interchangeably. Furthermore, we shall term a soft set in a soft space (X, E) as 'total soft set' if the soft set, which is a mapping, is defined on each point of the universe of attributes E . Hence (F, E) is a total soft set in the soft space (X, E) , but (G, B) , with $B \subset E$, is not.

It is noteworthy that the matrix representation compels one to write every soft set as a total soft set. Consequently neither matrix representation is unique, nor it returns the original soft set. This is shown by the following example:

Example 14. Let (X, E) be a soft space with $X = \{a, b, c\}$ and $E = \{e_1, e_2, e_3\}$. Choose

$$(F, A) = \{e_1 = \{a, c\}, e_3 = \{b, c\}\},$$

then its matrix representation is given as

$$F' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

If we try to retrieve, the soft set (F, A) from F' we get

$$(G, E) = \{e_1 = \{a, c\}, e_2 = \{\}, e_3 = \{b, c\}\},$$

which is clearly a different soft set in (X, E) as $(F, A) \subsetneq (G, E)$ but $(G, E) \not\subseteq (F, A)$ and hence $(F, A) \neq (G, E)$.

Moreover, it is evident by the very definition of soft union as given by Maji *et.al.* that total soft sets are not meant by either Molodstov [9] or Maji, Biswas and Roy [8]. Had this been the case, the soft union should not have been defined in three pieces.

Furthermore, it is important to note that in [7], while calculating the similarity only the value sets of a soft set have been paid attention to. Whereas, ideally a similarity measure for soft sets must reflect similarity between both the value sets and the attributes, due to the peculiar dependance of the notion of soft set upon these two sets.

Both the above given points viz. non-suitability of matrix representation and partial nature of similarity measures of Majumdar and Samanta, provide us motivation to introduce more suitable distance and similarity measures of soft sets. We introduce these measures in the following sections.

5. Distance Between Soft Sets

Recall that symmetric difference between two sets A and B is denoted and defined as:

$$A \Delta B = (A \cup B) \setminus (A \cap B).$$

We first define:

Definition 15. Let $(F, A), (G, B)$ and (H, C) be soft sets in a soft space (X, E) and $d : X \times X \rightarrow \mathbb{R}^+$ a mapping. Then

- (1) d is said to be quasi-metric if it satisfies
 - (M₁) $d((F, A), (G, B)) \geq 0$,
 - (M₂) $d((F, A), (G, B)) = d((G, B), (F, A))$,
- (2) A quasi-metric d is said to be semi-metric if
 - (M₃) $d((F, A), (G, B)) + d((G, B), (H, C)) \geq d((F, A), (H, C))$
- (3) A semi-metric d is said to be pseudo metric if
 - (M₄) $(F, A) = (G, B) \Rightarrow d((F, A), (G, B)) = 0$,
- (4) A pseudo metric d is said to be metric if
 - (M₅) $d((F, A), (G, B)) = 0 \Rightarrow (F, A) = (G, B)$.

Some quasi-metrics and semi-metrics for soft sets may readily be defined as follows:

Definition 16. For two soft sets (F, A) and (G, B) in a soft space (X, E) , where A and B are not identically void, we define Hamming quasi-metric as:

$$d((F, A), (G, B)) = \|A \Delta B\| + \sum_{\varepsilon \in A \cap B} \|F(\varepsilon) \Delta G(\varepsilon)\|,$$

and Normalized Hamming quasi-metric as:

$$l((F, A), (G, B)) = \frac{\|A \Delta B\|}{\|A \cup B\|} + \sum_{\varepsilon \in A \cap B} \chi(\varepsilon)$$

$$\text{where } \chi(\varepsilon) = \begin{cases} \frac{\|F(\varepsilon) \Delta G(\varepsilon)\|}{\|F(\varepsilon) \cup G(\varepsilon)\|}, & \text{if } F(\varepsilon) \cup G(\varepsilon) \neq \phi \\ 0, & \text{otherwise.} \end{cases}$$

Definition 17. For two soft sets (F, A) and (G, B) in a soft space (X, E) , we define Cardinality semi-metric as:

$$c((F, A), (G, B)) = |||A|| - ||B|| + \sum_{\varepsilon \in A \cap B} |||F(\varepsilon)|| - ||G(\varepsilon)|||,$$

and Normalized Cardinality semi-metric as:

$$p((F, A), (G, B)) = \frac{|||A|| - ||B|||}{\|E\|} + \sum_{\varepsilon \in A \cap B} \frac{|||F(\varepsilon)|| - ||G(\varepsilon)|||}{\|X\|}.$$

Following example shows that d and l are quasi-metrics and c and p are semi-metrics, only:

Example 18. Let (X, E) be a soft space with $X = \{a, b, c, d\}$ and $E = \{e_1, e_2, e_3, e_4\}$. We choose following soft sets in (X, E) :

$$\begin{aligned} (F, A) &= \{e_1 = \{c, b\}, e_2 = \{b\}, e_3 = \{a, b, c\}, e_4 = \{d\}\}, \\ (G, B) &= \{e_2 = \{b, c\}, e_3 = \{a, b, c, d\}\}, \\ (H, C) &= \{e_1 = \{b, d\}, e_2 = \{b, c, d\}, e_3 = \{a, d\}, e_4 = \{a, b, c, d\}\}. \end{aligned}$$

Then calculations give

$$\begin{aligned} d((F, A), (G, B)) &= 4 & d((G, B), (H, C)) &= 5 & d((F, A), (H, C)) &= 10 \\ l((F, A), (G, B)) &= \frac{5}{4} & l((G, B), (H, C)) &= \frac{4}{3} & l((F, A), (H, C)) &= \frac{17}{6} \end{aligned}$$

and hence

$$\begin{aligned} d((F, A), (G, B)) + d((G, B), (H, C)) &= 4 + 5 \not\geq 10 = d((F, A), (H, C)), \\ l((F, A), (G, B)) + l((G, B), (H, C)) &= \frac{5}{4} + \frac{4}{3} \not\geq \frac{17}{6} = l((F, A), (H, C)). \end{aligned}$$

Again choose $X = \{a, b, c\}$, $E = \{e_1, e_2, e_3, e_4\}$ and soft sets:

$$\begin{aligned} (F, A) &= \{e_3 = \{c\}, e_4 = \{a\}, e_1 = \{c, b, a\}, e_2 = \{b, a\}\}, \\ (G, B) &= \{e_3 = \{c, b\}, e_4 = \{c, b\}, e_2 = \{b, a\}\}, \\ (H, C) &= \{e_4 = \{c, b, a\}, e_1 = \{c\}, e_2 = \{b, a\}\}. \end{aligned}$$

Then we get

$$\begin{aligned} c((F, A), (G, B)) + c((G, B), (H, C)) &= 3 + 1 \not\geq 5 = c((F, A), (H, C)), \\ p((F, A), (G, B)) + p((G, B), (H, C)) &= \frac{11}{12} + \frac{1}{3} \not\geq \frac{19}{12} = p((F, A), (H, C)). \end{aligned}$$

Moreover c and p fail to satisfy

$$\begin{aligned} c((F, A), (G, B)) &= 0 \iff (F, A) = (G, B), \\ p((F, A), (G, B)) &= 0 \iff (F, A) = (G, B). \end{aligned}$$

For this choose

$$\begin{aligned} (F, A) &= \{e_4 = \{b, a\}, e_1 = \{\}\}, \\ (G, B) &= \{e_3 = \{b\}, e_4 = \{c, b\}\}, \end{aligned}$$

then

$$c((F, A), (G, B)) = 0 = p((F, A), (G, B)).$$

Definition 19. For two soft sets (F, A) and (G, B) in a soft space (X, E) , where A and B are not identically void, we define Euclidean distance as:

$$e((F, A), (G, B)) = \|A \Delta B\| + \sqrt{\sum_{\varepsilon \in A \cap B} \|F(\varepsilon) \Delta G(\varepsilon)\|^2},$$

Normalized Euclidean distance as:

$$\begin{aligned} q((F, A), (G, B)) &= \frac{\|A \Delta B\|}{\sqrt{\|A \cup B\|}} + \sqrt{\sum_{\varepsilon \in A \cap B} \chi(\varepsilon)}. \\ \text{where } \chi(\varepsilon) &= \begin{cases} \frac{\|F(\varepsilon) \Delta G(\varepsilon)\|^2}{\|F(\varepsilon) \cup G(\varepsilon)\|}, & \text{if } F(\varepsilon) \cup G(\varepsilon) \neq \phi \\ 0, & \text{otherwise} \end{cases}, \end{aligned}$$

where all the radicals yield non-negative values only.

Proposition 20. The mappings $e, q : (X, E) \times (X, E) \rightarrow \mathbb{R}^+$, as defined above, are metrics.

Lemma 21. For the soft sets $(F_\phi, E), (F_X, E)$ and an arbitrary soft set (F, A) in a soft space (X, E) , we have:

- (1) $e((F, A), (F, A)^c) = 2\|A\|,$
- (2) $q((F, A), (F, A)^c) = \sqrt{2\|A\|},$
- (3) $e((F_\phi, E), (F_X, E)) = \sqrt{\|E\|\|X\|},$
- (4) $q((F_\phi, E), (F_X, E)) = \sqrt{\|E\|}.$

6. Some New Similarity Measures

Definition 22. A mapping $S : (X, E) \times (X, E) \rightarrow [0, 1]$ is said to be similarity measure if its value $S((F, A), (G, B))$, for arbitrary soft sets (F, A) and (G, B) in the soft space (X, E) , satisfies following axioms:

- (s1) $0 \leq S((F, A), (G, B)) \leq 1,$
- (s2) if $(F, A) = (G, B)$, then $S((F, A), (G, B)) = 1,$
- (s3) $S((F, A), (G, B)) = S((G, B), (F, A)),$
- (s4) if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (H, C)$, then $S((F, A), (H, C)) \leq S((F, A), (G, B))$ and $S((F, A), (H, C)) \leq S((G, B), (H, C)).$

Definition 23. For two soft sets (F, A) and (G, B) in a soft space (X, E) , we define a set theoretic matching function similarity measure as:

$$M((F, A), (G, B)) = \frac{\|A \cap B\|}{\max(\|A\|, \|B\|)} + \frac{\sum_{\varepsilon \in A \cap B} \|F(\varepsilon) \cap G(\varepsilon)\|}{\sum_{\varepsilon \in A \cap B} \max(\|F(\varepsilon)\|, \|G(\varepsilon)\|)}.$$

Proposition 24. For the soft sets $(F_\phi, E), (F_X, E)$ and an arbitrary soft set (F, A) in a soft space (X, E) , we have:

- (1) $M((F, A), (F, A)^c) = 0,$
- (2) $M((F_\phi, E), (F_X, E)) = 1.$

Based upon distances, defined in last section (Definition 19), two similarity measure may be introduced, following Koczy [4], as:

$$S_K^e((F, A), (G, B)) = \frac{1}{1 + e((F, A), (G, B))},$$

$$S_K^q((F, A), (G, B)) = \frac{1}{1 + q((F, A), (G, B))}.$$

Using the definition of Williams and Steele [12] we may define another pair of similarity measures as:

$$S_W^e((F, A), (G, B)) = e^{-\alpha \cdot e((F, A), (G, B))},$$

$$S_W^q((F, A), (G, B)) = e^{-\alpha \cdot q((F, A), (G, B))}.$$

where α is a positive real number (parameter) called the steepness measure.

Definition 25. [7] Two soft sets (F, A) and (G, B) in a soft space (X, E) are said to be α -similar, denoted as $(F, A) \overset{\alpha}{\approx} (G, B)$, if

$$S((F, A), (G, B)) \geq \alpha \text{ for } \alpha \in (0, 1),$$

where S is a similarity measure.

Proposition 26. $\overset{\alpha}{\approx}$ is reflexive and symmetric.

Majumdar and Samanta [7] have defined the notion of significant similarity as follows:

Definition 27. [7] Two soft sets (F, A) and (G, B) in a soft space (X, E) are said to be significantly similar with respect to the similarity measure S , if $S((F, A), (G, B)) \geq \frac{1}{2}$.

In the following example we show that two clearly non-similar soft sets come out to be significantly similar using a Majumdar-Samant similarity measure. But the same soft sets are rightly discerned as non-significantly similar by a similarity measure proposed in this work:

Example 28. Let $X = \{a, b, c, d\}$ and $E = \{e_1, e_2, e_3\}$ and

$$\begin{aligned} (F, A) &= \{e_1 = \{\}, e_2 = \{\}\}, \\ (G, B) &= \{e_2 = \{b, d\}\}, \\ (H, C) &= \{e_2 = \{b, c, d\}\}. \end{aligned}$$

It is intuitively clear that (F, A) and (H, C) are not similar but (G, B) and (H, C) appear to be considerably similar. We calculate the similarity of both pairs of soft sets using a Majumdar-Samanta similarity measure $S'(F_1, F_2) = \frac{1}{1+E^S(F_1, F_2)}$, as follows:

$$\begin{aligned} S'((F, A), (H, C)) &= \frac{1}{2} = 0.5, \\ S'((G, B), (H, C)) &= \frac{1}{1 + \frac{\sqrt{3}}{3}} = 0.633. \end{aligned}$$

Hence according to S' both the soft sets (F, A) and (G, B) are significantly similar to (H, C) , though this conclusion is counter-intuitive. On the other hand using S_K^e (proposed in this work) we calculate similarities as:

$$\begin{aligned} S_K^e((F, A), (H, C)) &= \frac{1}{2 + \sqrt{3}} = 0.25, \\ S_K^e((G, B), (H, C)) &= \frac{1}{2} = 0.5. \end{aligned}$$

Clearly S_K^e has rightly discerned $(F, A), (H, C)$ to be non-significantly similar and $(G, B), (H, C)$ as significantly similar.

We now give some interesting properties of the newly introduced similarity measures in the form of following two propositions. Proofs of these propositions are straightforward in view of Lemma 21:

Proposition 29. For an arbitrary soft set (F, A) in a soft space (X, E) , we have:

- (1) $S_K^e((F, A), (F, A)^c) = \frac{1}{1+2\|A\|}$,
- (2) $S_K^q((F, A), (F, A)^c) = \frac{1}{1+\sqrt{2\|A\|}}$,
- (3) $S_W^e((F, A), (F, A)^c) = e^{-2\|A\|^\alpha}$,
- (4) $S_W^q((F, A), (F, A)^c) = e^{-\sqrt{2\|A\|}^\alpha}$.

Proposition 30. For the soft sets (F_ϕ, E) , (F_X, E) in a soft space (X, E) , we have:

- (1) $S_K^e(((F_\phi, E), (F_X, E))) = \frac{1}{1+\sqrt{\|E\|\|X\|}}$,
- (2) $S_K^q(((F_\phi, E), (F_X, E))) = \frac{1}{1+\sqrt{\|E\|}}$,
- (3) $S_W^e(((F_\phi, E), (F_X, E))) = e^{-\sqrt{\|E\|\|X\|}^\alpha}$,
- (4) $S_W^q(((F_\phi, E), (F_X, E))) = e^{-\sqrt{\|E\|}^\alpha}$.

7. An Application of Similarity Measures in Financial Diagnosis

We now present a financial diagnosis problem where similarity measures can be applied.

The notion of similarity measure of two soft sets can be applied to detect whether a firm is suffering from a certain economic syndrome or not. In the following example, we estimate if two firms with observed profiles of financial indicators are suffering from serious liquidity problem. Suppose the firm profiles are given as:

Profile 1 The firm ABC maintains a beerish future outlook as well as same behaviour in trading of its share prices. During last fiscal year the profit-earning ratio continued to rise. Inflation is increasing continuously. ABC has a low amount of paid-up capital and a similar situation is seen in foreign direct investment flowing into ABC.

Profile 2 The firm XYZ showed a fluctuating share price and hence a varying future outlook. Like ABC profit-earning ratio remained beerish. As both firms are in the same economy, inflation is also rising for XYZ and may be considered even high in view of XYZ. Competition in the business area of XYZ is increasing. Debit level went high but the paid-up capital lowered.

For this, we first construct a model soft set for liquidity-problem and the soft sets for the firm profiles. Next we find the similarity measure of these soft sets. If they are significantly similar, then we conclude that the firm is possibly suffering from liquidity problem.

Let $X = \{\text{inflation, profit-earning ratio, share price, paid-up capital, competitiveness, business diversification, future outlook, debt level, foreign direct investment, fixed income}\}$ be the collection of financial indicators which are given in both profiles. Further let $E = \{\text{fluctuating, medium, rising, high, beerish}\}$ be the universe of parameters, which are basically linguistic labels commonly used to describe the state of financial indicators.

The profile of a firm by observing its financial indicators may easily be coded into a soft set using appropriate linguistic labels. Let (F, A) and (G, B) be soft sets coding

profiles of firms ABC and XYZ, respectively, and are given as:

$$\begin{aligned} (F, A) &= \left\{ \begin{array}{l} \text{bearish} = \{\text{future outlook, share price}\}, \text{ rising} = \{\text{profit earning ratio, inflation}\}, \\ \text{low} = \{\text{paid-up capital, foreign direct investment}\} \end{array} \right\}, \\ (G, B) &= \left\{ \begin{array}{l} \text{fluctuating} = \{\text{share price, future outlook}\}, \text{ beerish} = \{\text{profit earning ratio}\}, \\ \text{rising} = \{\text{inflation, compitition}\}, \text{ high} = \{\text{inflation, debit level}\}, \text{ low} = \{\text{paid-up capital}\} \end{array} \right\}. \end{aligned}$$

The model soft set for a firm suffering from liquidity problem can easily be prepared in a similar manner by help of a financial expert. In our case we take it to be as follows:

$$(H, C) = \left\{ \begin{array}{l} \text{fluctuating} = \{\text{share price, future outlook}\}, \text{ low} = \{\text{fixed income, paid-up capital}\} \\ \text{beerish} = \{\text{profit earning ratio, foreign direct investment}\}, \text{ high} = \{\text{inflation, debt level}\} \end{array} \right\}.$$

For the sake of ease in mathematical manipulation we denote the indicators and labels by symbols as follows:

i	=	inflation		
p	=	profit-earning ratio		
s	=	share price		
c	=	paid-up capital	e_1	= fluctuating
m	=	competition	e_2	= low
d	=	business diversification	and e_3	= rising
o	=	future outlook	e_4	= high
l	=	debt level	e_5	= beerish
f	=	foreign direct investment		
x	=	fixed income		

Thus we have $X = \{i, p, s, c, m, d, o, l, f, x\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$ and the soft sets of firm profiles become:

$$\begin{aligned} (F, A) &= \{e_5 = \{o, s\}, e_3 = \{p, i\}, e_2 = \{s, f\}\}, \\ (G, B) &= \{e_1 = \{s, o\}, e_2 = \{c\}, e_3 = \{i, m\}, e_4 = \{i, l\}, e_5 = \{p, f\}\}, \\ (H, C) &= \{e_1 = \{o, s\}, e_2 = \{c\}, e_4 = \{i, l\}, e_5 = \{p, f\}\}. \end{aligned}$$

As the calculations give:

$$\begin{aligned} S_K^e((F, A), (H, C)) &= \frac{1}{4 + \sqrt{7}} = 0.15, \\ S_K^e((G, B), (H, C)) &= \frac{1}{2} = 0.5. \end{aligned}$$

Hence we conclude that the firm with profile (G, B) i.e. XYZ is suffering from a liquidity problem as its soft set profile is significantly similar to the standard liquidity problem profile. Whereas the firm ABC is very less likely to be suffering from the same problem.

Conclusion 31. Majumdar and Samanta [7] use matrix representation based distances of soft sets to introduce matching function and distance based similarity measures. We first give counterexamples to show that Majumdar and Samanta's Definition 2.7 and Lemma 3.5(3) contain errors, then prove some properties of the distances introduced by them, thus making their Lemma 4.4, a corollary of our result.

The tacit assumption of [7] that matrix representation is a suitable representation for mathematical manipulation of soft sets, has been shown to be flawed, in Section 4. This raises a natural question as to what approach be considered suitable for similarity measures of soft sets? In one possible reply to this we introduce set operations based measures. Our Example 28 presents a case where Majumdar-Samanta similarity measure produces an erroneous result but the measure proposed herein decides correctly. The new similarity measures have been applied to the problem of financial diagnosis of firms. A technique of using linguistic labels as parameters for soft sets has been used to model natural-language descriptions in terms of soft sets. This exhibits the rich prospects held by Soft Set Theory as a tool for problems in social, biological and economic systems.

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